UCLA

Math for Plasma Sciences

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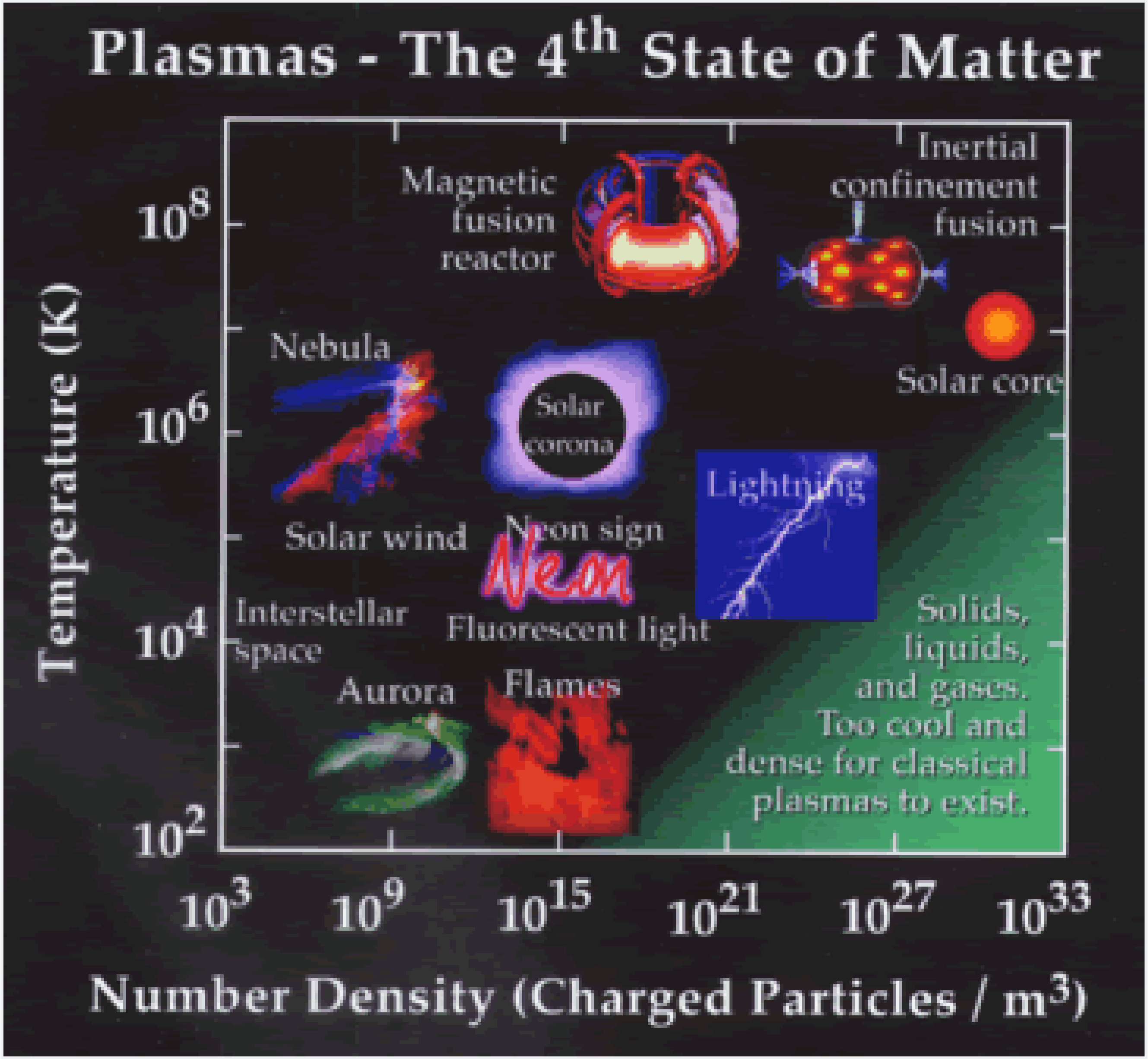
math changes everything.

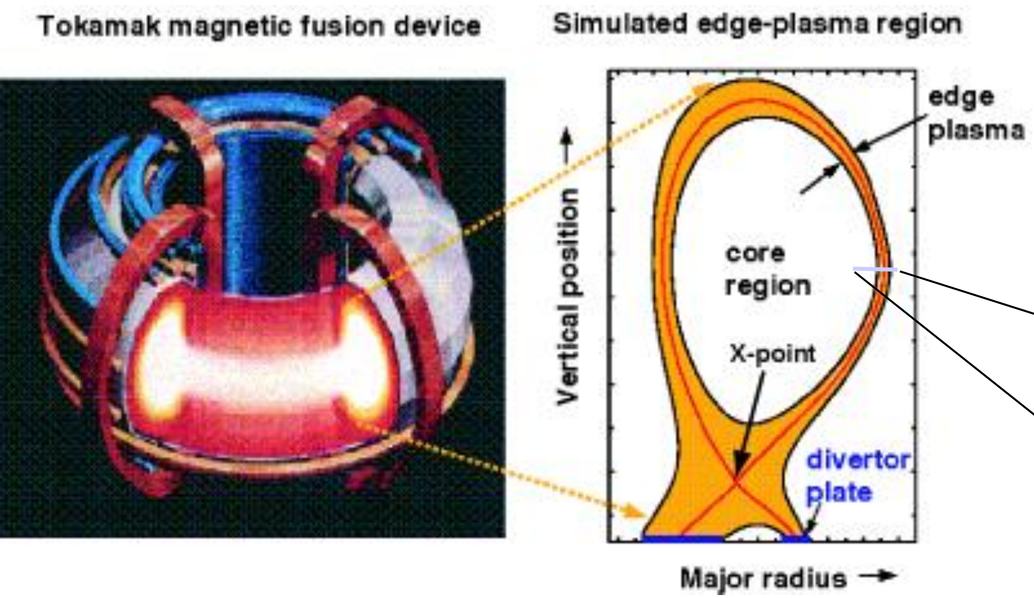
Plasma Fest 2015

Plasmas

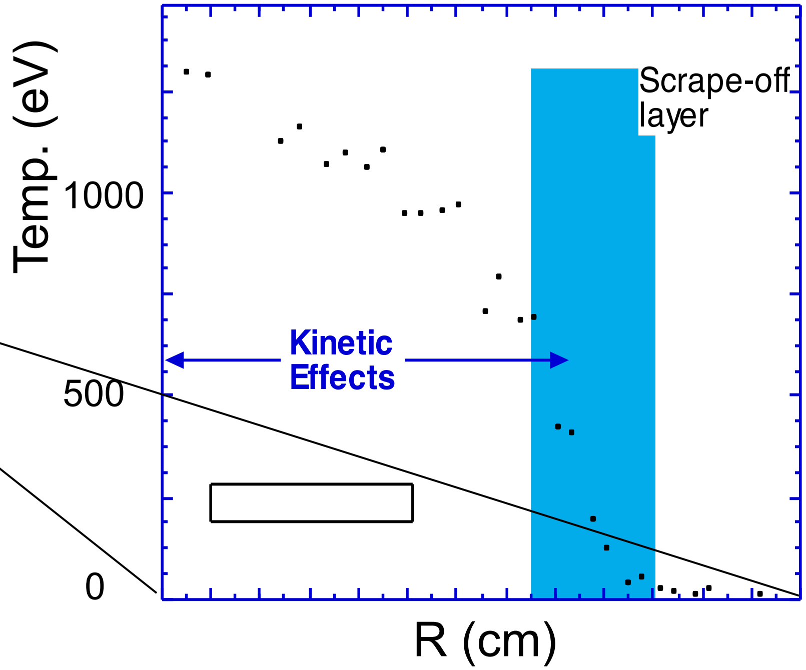


Plasmas - The 4th State of Matter
Temperature (K)
10^8
10^6
10^4
10^2
Magnetic fusion reactor
Inertial confinement fusion
Nebula
Solar corona
Solar core
Lightning
Solar wind
Neon sign
Interstellar space
Fluorescent light
Aurora
Flames
Solids, liquids, and gases. Too cool and dense for classical plasmas to exist.
10^3
10^9
10^{15}
10^{21}
10^{27}
10^{33}
Number Density (Charged Particles / m^3)

Where are collisions signifiant in plasmas?
Example: Tokamak edge boundary layer



Schematic views of divertor tokamak and edge-plasma region (magnetic separatrix is the red line and the black boundaries indicate the shape of magnetic flux surfaces)



Edge pedestal temperature profile near the edge of an H-mode discharge in the DIII-D tokamak. [Porter*2000*]. Pedestal is shaded region.

What Can Math Add?

- Phenomenology
  - Multiscale
  - Complex
- Numerical methods
  - Monte Carlo simulation
- New mathematical methods
  - Compressed sensing and sparsity

Examples of Math for Plasmas

- Hybrid combination of continuum and particle dynamics
- Sparsity and compressed sensing

Accelerated Simulation for Plasma Kinetics

- Hybrid schemes
- Negative particles

Kinetic Theory

- Phase space particle number density $f(\vec{x}, \vec{v}, t)$
- Boltzmann Equation:

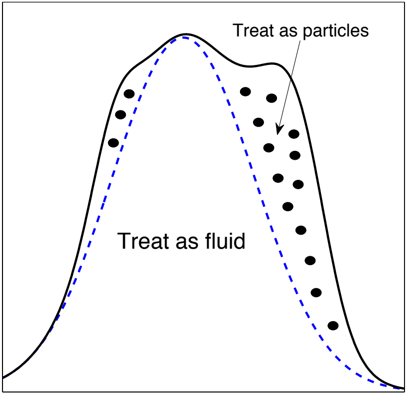
$$\partial_t f + \vec{v} \cdot \nabla_x f + \vec{a} \cdot \nabla_v f = Q(f,f) \tag{1}$$

$Q(f,f)$ is:

- Boltzmann collision operator for rarefied gas dynamics (RGD)
- Landau-Fokker-Planck (LFP) operator for Coulomb collisions

Hybrid Scheme

Combine fluid and particle simulation methods[1]:

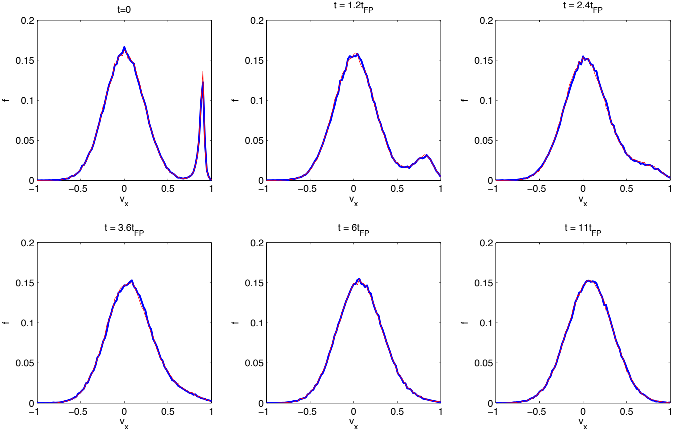


- Separate f into Maxwellian and non-Maxwellian components: $f = m + k$
- Treat m as fluid
- Simulate k by particles
- Sample particles from m and collide them with k
- Fully nonlinear δf
- Thermalize particles in k, using entropy criterion[a]

[a] Ricketson et. al, J. Comp. Phys., 2014.

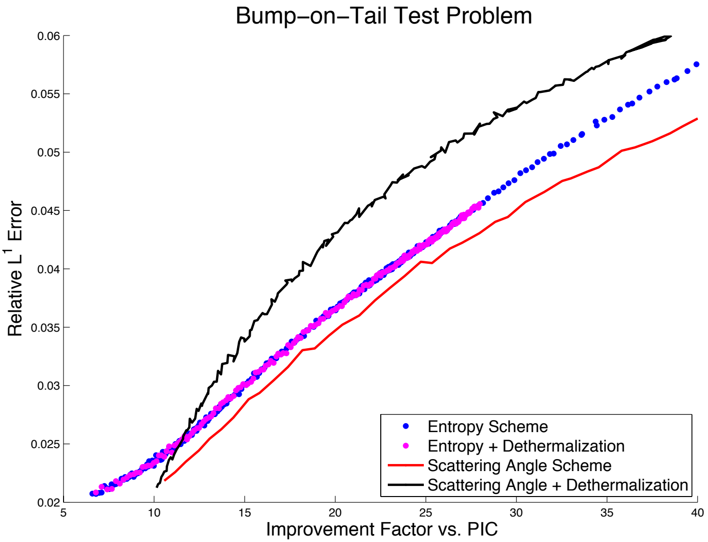
[1] Caflisch et. al, Multiscale Model. Simul. 2008

Bump-on-Tail



t=0
t = 1.2t_FP
t = 2.4t_FP
t = 3.6t_FP
t = 6t_FP
t = 11t_FP
f
v_x
0.2
0.15
0.1
0.05
0
−1
−0.5
0
0.5
1

Scheme Comparison



Negative particles

Negative deviations from Maxwellian through $k = f_p - f_n$

- Particle number can increase
    - Offsetting pairs of positive and negative particles
- Reformulation that limits growth of particles[2]
    - Coarse-grained, direct simulation

$$\partial_t \tilde{f} = Q(\tilde{f}, \tilde{f})$$

    - Fine-grained, deviatoric simulation

$$\begin{aligned} \partial_t m &= 0 \\ \partial_t f_p &= Q(\tilde{f}, f_p) + Q(f_p - f_n, m)_+ \\ \partial_t f_n &= Q(\tilde{f}, f_n) + Q(f_p - f_n, m)_- \end{aligned}$$

    - Limits growth of particle number

[2] Yan & Caflisch, J. Comp. Phys., to appear.

Nonlinear Landau damping

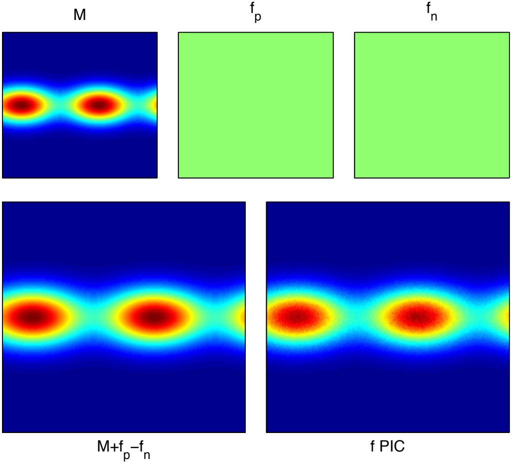


Figure: The snapshot in phase space (x, v_1). $t = 0.0000$.

Nonlinear Landau damping

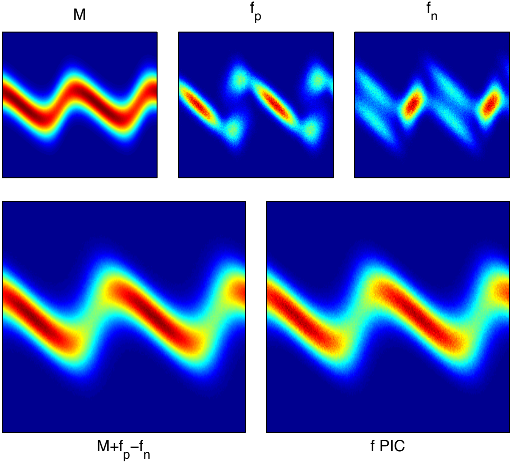


Figure: The snapshot in phase space (x, v_1). $t = 0.5236$.

Nonlinear Landau damping

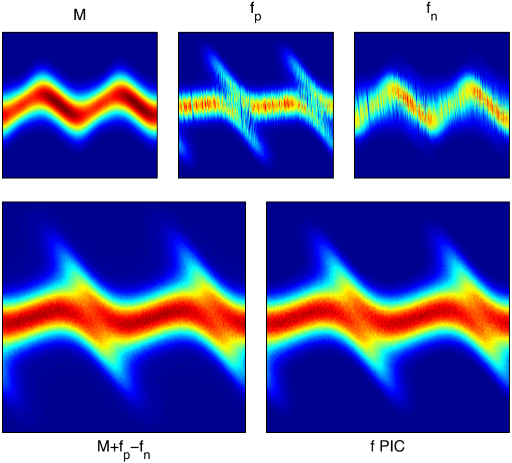


Figure: The snapshot in phase space (x, v_1). $t = 1.1519$.

Compressed Sensing and Sparsity

- Developed for information science and optimization
- Applications to PDEs and physics are starting
- Possible applications to plasma science
  - Reduced description of complex phenomena
  - Framework for multiscale modeling and simulation

Sparsity

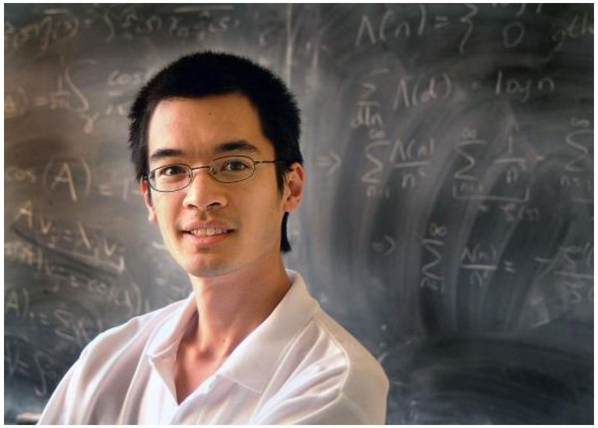
- Sparsity in datasets (e.g., sensor signals)
  - Signal $x \in R^N$ which is "m-sparse", with $m << N$
    - i.e., x has at most m non-zero components
  - n measurements of x, corresponds to

$$f = Ax \in R^n \quad A \text{ is } n \times N$$

- Objectives
  - How many measurements are required?
    - What is the value of n?
  - How hard is it to compute x?
    - Tractable or intractable?

Compressed Sensing

- Compressed sensing 2006
  - David Donoho
  - Emmanuel Candes, Justin Romberg & Terry Tao



How many measurements are required?

- For m << N, find m-sparse solution $x \in R^N$ of

$$Ax = f \in R^n \qquad A \; is \; n \times N$$

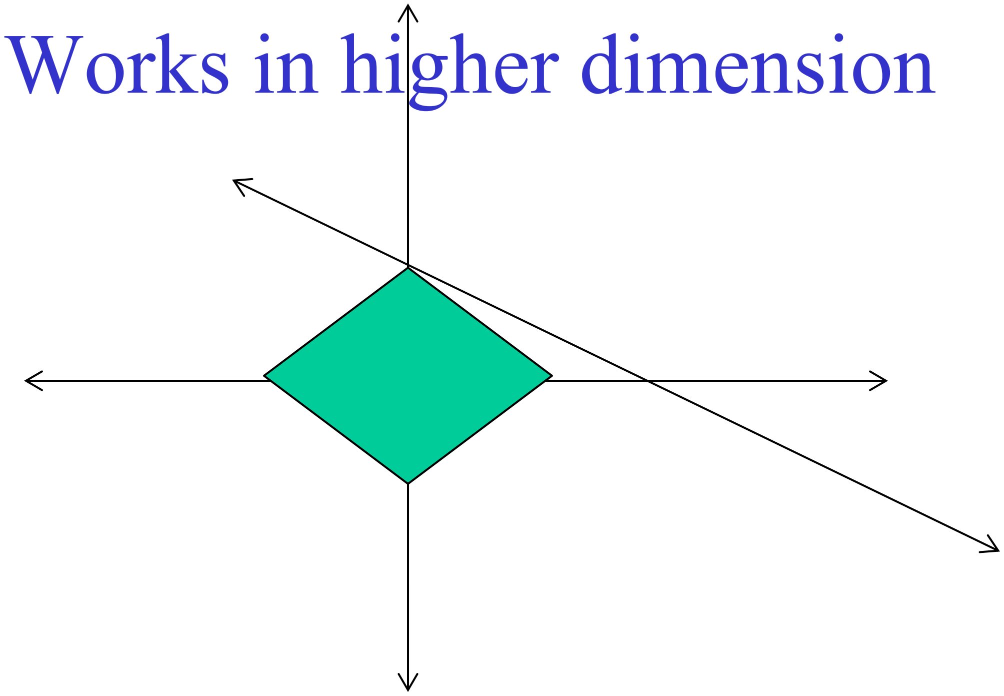
- Standard methods require: n = N
  - #(equations)=#(unknowns)
  - Np hard
- Compressed sensing: n = m (log N)
  - n << N. Many fewer equations than unknowns!
  - Solution is exact with high probability!
    - Reduced isometry property (RIP)
  - convex programming

Geometric description

- Find x on line $a_1x_1 + a_2x_2 = f$ with smallest $\|x\|_1$

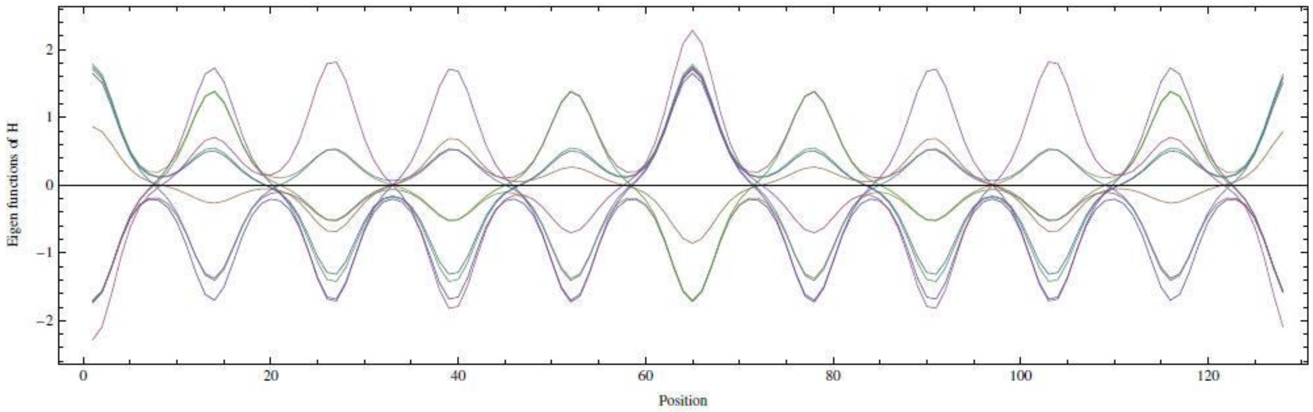
$$\|x\|_1 = |x_1| + |x_2|$$

  – For all but 45° lines, L^1 norm is smallest at a vertex.
- Vertices are sparse points, since a component is 0.
- Works in higher dimension

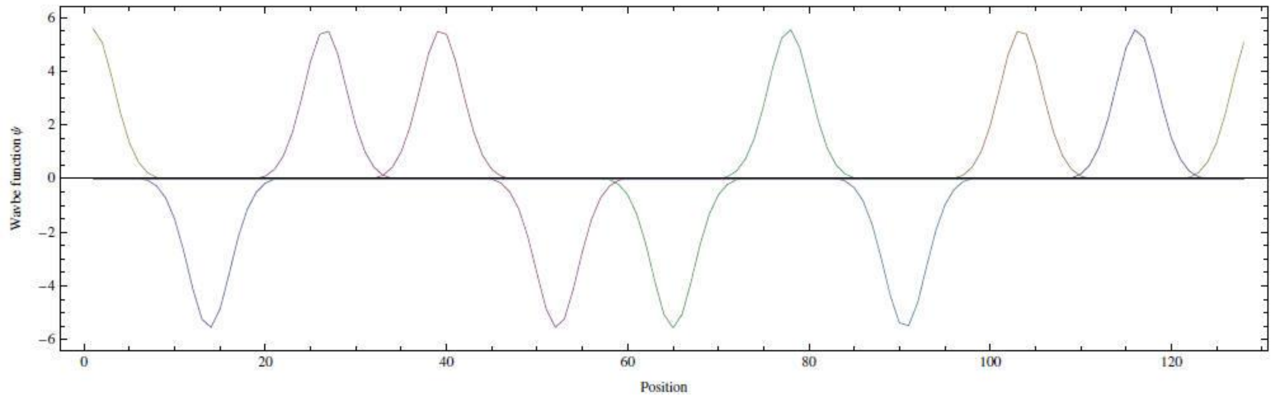


Eigen-Modes vs. Compressed Modes for 1D Kronig-Penney

10 Eigen-Modes for KP



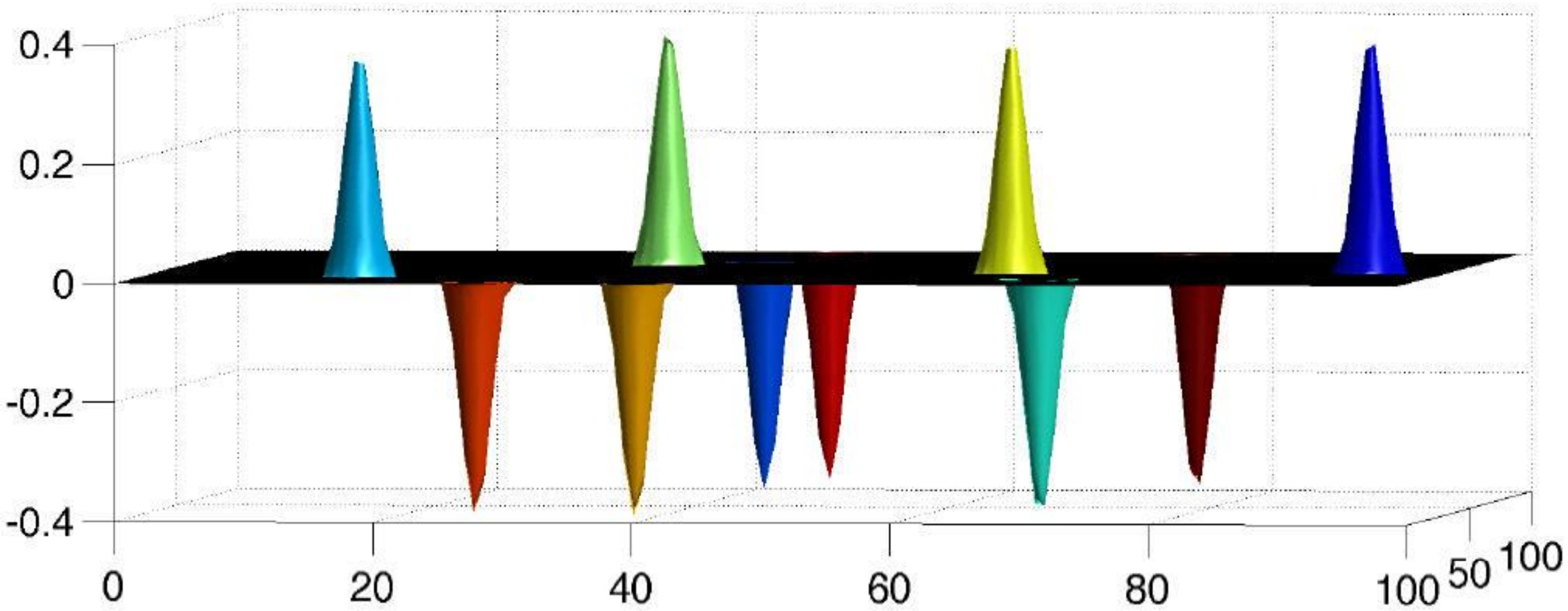
10 Compressed Modes for KP



Using SOC algorithm (Lai, Osher) for Bregman iteration

Ozolins, Lai, Caflisch & Osher, PNAS 2013

Compressed Modes for 3D Free Electron Model (V=0)



Mathematicians at Plasma Fest

- UCLA

Faculty

Chris Anderson
Russ Caflisch
Stan Osher
Joey Teran

Student & Postdocs

Yi Chen
Hai Le
Craig Schroeder
Jonathan Siegel
Bokai Yan

- LLNL

Jeff Hittinger

- AFRL

Rob Martin

- UC Merced

Maya Tokman