Math for Plasma Sciences

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math changes everything.

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Plasmas - The 4th State of Matter

Temperature (K)

10^8
10^6
10^4
10^2

Number Density (Charged Particles / m^3)

10^3
10^9
10^15
10^21
10^27
10^33

Magnetic fusion reactor
Inertial confinement fusion
Solar core

Nebula
Solar wind
Interstellar space
Aurora
Flames

Solar corona
Fluorescent light
Neon sign
Lightning

Solids, liquids, and gases. Too cool and dense for classical plasmas to exist.

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Where are collisions significant in plasmas?
Example: Tokamak edge boundary layer

Schematic views of divertor tokamak and edge-plasma region (magnetic separatrix is the red line and the black boundaries indicate the shape of magnetic flux surfaces)

Edge pedestal temperature profile near the edge of an H-mode discharge in the DIII-D tokamak. [Porter2000]. Pedestal is shaded region.

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What Can Math Add?

• **Phenomenology**
  – Multiscale
  – Complex

• **Numerical methods**
  – Monte Carlo simulation

• **New mathematical methods**
  – Compressed sensing and sparsity
Examples of Math for Plasmas

- Hybrid combination of continuum and particle dynamics
- Sparsity and compressed sensing
Accelerated Simulation for Plasma Kinetics

- Hybrid schemes
- Negative particles
Phase space particle number density $f(\vec{x}, \vec{v}, t)$

Boltzmann Equation:

$$\partial_t f + \vec{v} \cdot \nabla_x f + \vec{a} \cdot \nabla_v f = Q(f,f) \quad (1)$$

$Q(f,f)$ is:

- Boltzmann collision operator for rarefied gas dynamics (RGD)
- Landau-Fokker-Planck (LFP) operator for Coulomb collisions
Hybrid Scheme

Combine fluid and particle simulation methods:\n
- Separate \( f \) into Maxwellian and non-Maxwellian components: \( f = m + k \)
- Treat \( m \) as fluid
- Simulate \( k \) by particles
- Sample particles from \( m \) and collide them with \( k \)
- Fully nonlinear \( \delta f \)
- Thermalize particles in \( k \), using entropy criterion\(^a\)

\(^a\)Ricketson et. al, J. Comp. Phys., 2014.

\(^1\)Caflisch et. al, Multiscale Model. Simul. 2008
Accelerated Simulation for Plasma Kinetics
Bump-on-Tail Test Problem

Relative $L^1$ Error vs. Improvement Factor vs. PIC

- Entropy Scheme
- Entropy + Dethermalization
- Scattering Angle Scheme
- Scattering Angle + Dethermalization

Accelerated Simulation for Plasma Kinetics
Negative deviations from Maxwellian through $k = f_p - f_n$

- Particle number can increase
  - Offsetting pairs of positive and negative particles
- Reformulation that limits growth of particles\(^2\)
  - Coarse-grained, direct simulation
    \[
    \partial_t \tilde{f} = Q(\tilde{f}, \tilde{f})
    \]
  - Fine-grained, deviatoric simulation
    \[
    \begin{align*}
    \partial_t m &= 0 \\
    \partial_t f_p &= Q(\tilde{f}, f_p) + Q(f_p - f_n, m)_+ \\
    \partial_t f_n &= Q(\tilde{f}, f_n) + Q(f_p - f_n, m)_-
    \end{align*}
    \]
- Limits growth of particle number

Nonlinear Landau damping

Figure: The snapshot in phase space \((x, v_1)\). \(t = 0.0000\).
Nonlinear Landau damping

Figure: The snapshot in phase space \((x, v_1)\). \(t = 0.5236\).
Nonlinear Landau damping

Figure: The snapshot in phase space $\{(x, v_1)\}$. $t = 1.1519$. 

Accelerated Simulation for Plasma Kinetics
Compressed Sensing and Sparsity

• Developed for information science and optimization
• Applications to PDEs and physics are starting
• Possible applications to plasma science
  – Reduced description of complex phenomena
  – Framework for multiscale modeling and simulation
Sparsity

• Sparsity in datasets (e.g., sensor signals)
  – Signal \( x \in \mathbb{R}^N \) which is “m-sparse”, with \( m << N \)
    • i.e., \( x \) has at most \( m \) non-zero components
  – \( n \) measurements of \( x \), corresponds to
    \[
    f = Ax \in \mathbb{R}^n \quad A \text{ is } n \times N
    \]

• Objectives
  – How many measurements are required?
    • What is the value of \( n \)?
  – How hard is it to compute \( x \)?
    • Tractable or intractable?
Compressed Sensing

- Compressed sensing 2006
  - David Donoho
  - Emmanuel Candes, Justin Romberg & Terry Tao
How many measurements are required?

• For \( m \ll N \), find \( m \)-sparse solution \( x \in \mathbb{R}^N \) of
  \[
  Ax = f \in \mathbb{R}^n \quad A \text{ is } n \times N
  \]

• Standard methods require: \( n = N \)
  – #(equations) = #(unknowns)
  – \( \text{NP hard} \)

• Compressed sensing: \( n = m (\log N) \)
  – \( n \ll N \). Many fewer equations than unknowns!
  – Solution is exact with high probability!
    • Reduced isometry property (RIP)
    – convex programming
Geometric description

- Find \( x \) on line \( a_1 x_1 + a_2 x_2 = f \) with smallest \( \|x\|_1 \)
  \[
  \|x\|_1 = |x_1| + |x_2|
  \]
  - For all but 45° lines, \( L^1 \) norm is smallest at a vertex.
- Vertices are sparse points, since a component is 0.
- Works in higher dimension

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Eigen-Modes vs. Compressed Modes for 1D Kronig-Penney

10 Eigen-Modes for KP

![Graph of 10 Eigen-Modes for KP](image)

10 Compressed Modes for KP

![Graph of 10 Compressed Modes for KP](image)

Using SOC algorithm (Lai, Osher) for Bregman iteration

Ozolins, Lai, Caflisch & Osher, PNAS 2013

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Compressed Modes
for 3D Free Electron Model (V=0)
Mathematicians at Plasma Fest

• UCLA
  Faculty
  Chris Anderson
  Russ Caflisch
  Stan Osher
  Joey Teran

Student & Postdocs
  Yi Chen
  Hai Le
  Craig Schroeder
  Jonathan Siegel
  Bokai Yan

• LLNL
  Jeff Hittinger

• AFRL
  Rob Martin

• UC Merced
  Maya Tokman